A201 problem set 1 Solutions

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1 M87 and a "Continuum" of Stars

a) The get the average separation distance between stars, d, put N = 10^{10} spheres inside the galactic sphere $R_g = 10$ kpc. The number density of stars is then

$$n = N/V = \frac{3N}{4\pi R_g^3} \tag{1}$$

The mean separation distance is given by $n^{-1/3}$ and is thus

$$d = R_g \left(\frac{4\pi}{3N}\right)^{1/3} = 7.4 \text{ pc} \tag{2}$$

The angular size, in radians, is given in the small angle approximation as $\theta = d/D$ where $D = 16$ Mpc is the distance to M87. Googling "radians to arcseconds" we find the transformation 1 radian $\simeq$ $206,265$ arcseconds, which gives

$$\theta = d/D \times 206,265 \simeq 0.1 \text{ arcsecond} \tag{3}$$

This may be resolvable with a good telescope like HST, so we will assume we are using a fairly poor ground based one.

b) The cross-section is the probability of interaction of light and matter between stars, which are both emitters and sources of scattering and absorption within the galaxy. The individual stars can be modeled as absorber spheres within the volume of the galaxy.

$$\sigma = \pi R_\odot^2 = 1.5 \times 10^{22} \text{ cm}^2 \tag{4}$$

The opacity is given by the following, where the average mass, $\overline{m}$, in this case is $M_\odot$:

$$\kappa = \frac{\sigma}{\overline{m}} = \frac{\pi R_\odot^2}{M_\odot} = 7.7 \times 10^{-12} cm^2/g \tag{5}$$

The following is the extinction coefficient:

$$\alpha = n\sigma = \rho\kappa = \frac{NM_\odot}{\frac{4}{3}\pi R_g^3}\frac{\pi R_\odot^2}{M_\odot} = \frac{3NR_\odot^2}{4R_g^3} = 1.2 \times 10^{-36} cm^{-1} \tag{6}$$

A typical value for opacity in astrophysics, e.g. ionized hydrogen, is $\kappa = 0.4$ cm^2/g. The opacity of stars is thus incredibly small in this context.

c) The optical depth is given by the following:

$$\tau_* = \alpha \Delta S = \alpha R_g = 3.7 \times 10^{-14} \ll 1 \tag{7}$$

This confirms that we can use the optically thin limit.

d) The total luminosity of the galaxy is $L = NL_\star$ where N is the number of stars and $L_\star$ the luminosity of each. The frequency integrated emission coefficient (grey j) of the stellar gas is then given by the following:

$$j = \frac{\text{Power}}{4\pi} = \frac{1}{4\pi}\frac{NL_\star}{\text{Volume}} = \frac{1}{4\pi}\frac{N4\pi R_\odot^2 \sigma T_\star^4}{\frac{4}{3}\pi R_g^3} == \frac{3}{4\pi}\frac{NR_\odot^2\sigma T_\star^4}{R_g^3} \tag{8}$$

Using the expression for α in the previous problem, we find that source function integrated over all frequencies is:

$$S = \frac{j}{\alpha} = \frac{\sigma T_\odot^4}{\pi} = B(T_\odot) \tag{9}$$

Where B is the frequency integrated Planck function. This is the same source function we would expect for a thermal emitter.

e) Integrate the specific intensity I(p) to determine the light profile of the galaxy. See Fig. 2 in problem set for geometry.

$$\frac{\partial I_\nu}{\partial s} = -\alpha I_\nu + j_\nu \tag{10}$$

Integrate both sides over frequency to get grey approximation.

$$\frac{\partial I}{\partial s} = -\alpha I + j \tag{11}$$

$$\frac{1}{\alpha}\frac{\partial I}{\partial s} = -I + S \tag{12}$$

where S = $\frac{j}{\alpha}$.

$$\int_{I_o}^{I} \frac{\mathrm{d}I}{I - S} = -\int_0^s \alpha \mathrm{d}s \tag{13}$$

$$\ln\left(\frac{I - S}{I_o - S}\right) = -\alpha s = -\tau \tag{14}$$

$$I = I_o e^{-\tau} - S(1 - e^{-\tau}) \tag{15}$$

Since we are working in the optically thin limit ($\tau \ll 1$) we can Taylor expand the exponential: $e^{-\tau} = 1 - \tau + ...$ Because there is no background source, I_o is zero when viewed from ∞. We then have the optically thin solution through a distance s of constant source function S

$$I = S\tau = S \times \alpha s = j \times s \tag{16}$$

From Pythagorean theorem, the length of the ray is $s = 2\sqrt{R_g^2 - p^2}$, where p is the impact parameter. We get:

$$I(p) = 2\alpha S\sqrt{R_g^2 - p^2} \tag{17}$$

In terms of the Planck function integrated over all frequencies:

$$S = B = \frac{1}{\pi}\sigma T^4 \tag{18}$$

$$I(p) = \frac{1}{\pi}2\alpha\sigma T^4\sqrt{R_g^2 - p^2} \tag{19}$$

f) Integrate I(p) to get the flux at Earth.

$$Flux = \oint I(p)\cos\theta d\Omega = -\oint 2j(R_g^2 - p^2)^{\frac{1}{2}}\mu d\mu d\phi \tag{20}$$

where d is distance to earth and p is the impact parameter, from geometry we get:

$$\mu = cos\theta = \frac{(d^2 - p^2)^{\frac{1}{2}}}{d^2} \tag{21}$$

and

$$d\mu = \frac{2p}{d^2}dp \tag{22}$$

Since we know $p \ll d$ we can to first order say $\mu = 1$. Then we have

$$F = -4\pi j\int_0^p (R_g^2 - p^2)^{\frac{1}{2}}\frac{p}{d^2}dp \tag{23}$$

$$F = \frac{4\pi j}{3d^2}(R_g^2 - p^2)^{\frac{3}{2}} = \frac{4\alpha\sigma T^4}{3d^2}(R_g^2 - p^2)^{\frac{3}{2}} \tag{24}$$

at p= 0, we have

$$F = \frac{4\alpha\sigma T^4}{3d^2}R_g^3 \tag{25}$$

The simple way of calculating the flux found by summing over the luminosities of every star and dividing by $4\pi d^2$ would give

$$F = \frac{NL_\odot}{4\pi d^2} = \frac{NR_\odot^2\sigma T^4}{d^2} \tag{26}$$

Comparing the two gives:

$$\frac{F_{integrated}}{F_{sum}} = \frac{4\alpha\sigma T^4 R_g^3}{3d^2}\frac{d^2}{NR_\odot^2\sigma T^4} = \frac{4R_g^3}{3NR_\odot^2}\alpha \tag{27}$$

$$\frac{F_{integrated}}{F_{sum}} = \frac{4R_g^3}{3NR_\odot^2}\frac{3NR_\odot^2}{4R_g^3} = 1 \tag{28}$$

So they are the same. This is true in this case because we assumed the trivial case of galaxy as an isotropic emitter. However, in general a source may be a non-isotropic emitter, in which case one really needs to do the integral to get the flux observed for any given viewing angle.

g) The estimated order of magnitude energy density near center is

$$u(r) = \frac{energy}{volume} = \frac{power * time}{volume} = \frac{NL_\odot}{\frac{4}{3}\pi R^3}\frac{R}{c} \approx 3\times 10^{-13}erg/cm^3 \tag{29}$$

while the flux near edge is

$$flux = \frac{NL_\odot}{4\pi R^2} \approx 3\times 10^{-3}erg/s/cm^2 \tag{30}$$

h) The geometry is given in Figure 1 the geometry. From the law of cosines

$$R^2 = r^2 + x^2 - 2rxcos(\pi - \theta) \tag{31}$$

$$x = -rcos\theta + \sqrt{R^2 - r^2sin^2\theta} \tag{32}$$

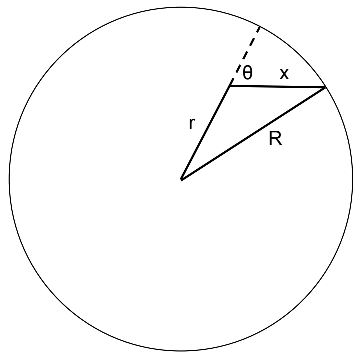


Figure 1: Geometry for 1h.

$$x = -rcos\theta + \sqrt{R^2 - r^2 sin^2\theta} \tag{33}$$

The specific intensity for an optically thin galaxy (using the same approximations as explained for equation 19) then becomes:

$$I_\tau = I_0 e^{-\tau} + S_\nu(1 - e^{-\tau}) = S_\nu \tau = S_\nu \alpha x = B_\nu \alpha x \tag{34}$$

$$I(r) = B_\nu \alpha (rcos\theta + \sqrt{R^2 - r^2 sin^2\theta}) \tag{35}$$

The mean intensity is found as follows:

$$J_\nu = \frac{B_\nu \alpha}{4\pi} \int\int I_\nu d\Omega = \frac{B_\nu \alpha}{4\pi} \int_0^{2\pi} \int_0^{\pi} (rcos\theta sin\theta + \sqrt{R^2 - r^2 sin^2\theta}) sin\theta d\theta d\phi \tag{36}$$

A 2π comes out of the phi integration, the first term in the integral disappears by symmetry (odd powers of cosθ), and we can simplify the second term by taking r << R:

$$J_\nu = \frac{B_\nu \alpha}{2} \int_0^{\pi} R(1 - \frac{1}{2}\frac{r^2}{R^2} sin^2\theta) sin\theta d\theta \tag{37}$$

$$J_\nu = \frac{B_\nu \alpha}{2} [2R - \frac{r^2}{2R} \int_0^{\pi} sin^3\theta d\theta] \tag{38}$$

$$J_\nu = \frac{B_\nu \alpha}{2} (2R - \frac{2r^2}{3R}) \tag{39}$$

$$J_\nu = B_\nu \alpha (R - \frac{r^2}{3R}) \tag{40}$$

The energy density is given by:

$$u_\nu = \frac{4\pi}{c} J_\nu = \frac{4\pi}{c} B_\nu \alpha (R - \frac{r^2}{3R}) \tag{41}$$

The flux is given by

$$F_\nu = \int I_\nu cos\theta d\Omega \tag{42}$$

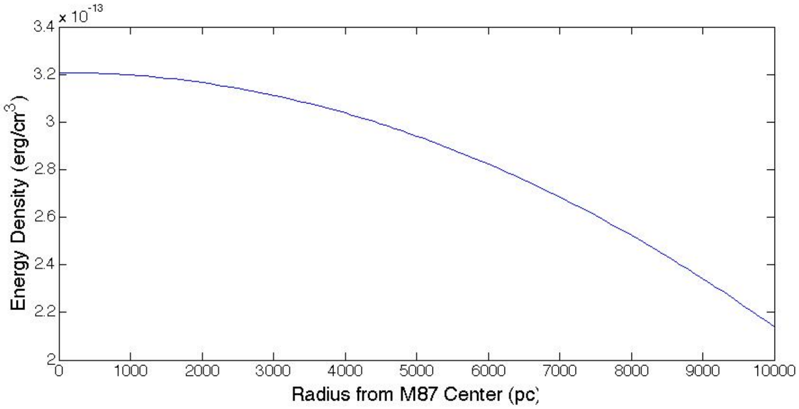


Figure 2: Energy density as a function of radius (in parsecs)

$$F_\nu = B_\nu \alpha \int_0^{2\pi} \int_0^{\pi} (rcos\theta + \sqrt{R^2 - r^2 sin^2\theta}) cos\theta sin\theta d\theta d\phi \quad (43)$$

$$F_\nu = 2\pi B_\nu \alpha \int_0^{\pi} (rcos^2\theta sin\theta + \sqrt{R^2 - r^2 sin^2\theta} cos\theta sin\theta) d\theta \quad (44)$$

The second term goes to zero:

$$F_\nu = 2\pi B_\nu \alpha \int_0^{\pi} rcos^2\theta sin\theta d\theta = 2\pi B_\nu \alpha r (\frac{2}{3}) \quad (45)$$

$$F_\nu = \frac{4\pi}{3} B_\nu \alpha r \quad (46)$$

1.1 i.

Plots are shown in Figures 2 and 3 where

$$B = \frac{\sigma T_\odot^4}{\pi} \quad (47)$$

Because the source is isotropic and spherically symmetric, the flux at any given point is the vector sum from all directions. At the center of the galaxy, an observer would see the flux from all of the stars cancels out by this symmetry. Moving away from center, the flux from behind the observer becomes larger than the flux from the stars in front of the observer, yielding a net flux > 0. This net flux increases at larger radii. The energy density, however, is most concentrated at the center. This can be illustrated by the following: If we take an imaginary volume equal to the volume of the galaxy and center it within the galaxy, it encloses the total energy of all of the stars within the galaxy. As the center of our imaginary, fixed volume moves away from the galactic center, it begins to enclose empty space, thereby decreasing the amount of enclosed energy.

1.2 j.

$$u_*(r) = aT^4 \quad (48)$$

where r is distance from M87 to Earth, T $\approx$ 1.8 K

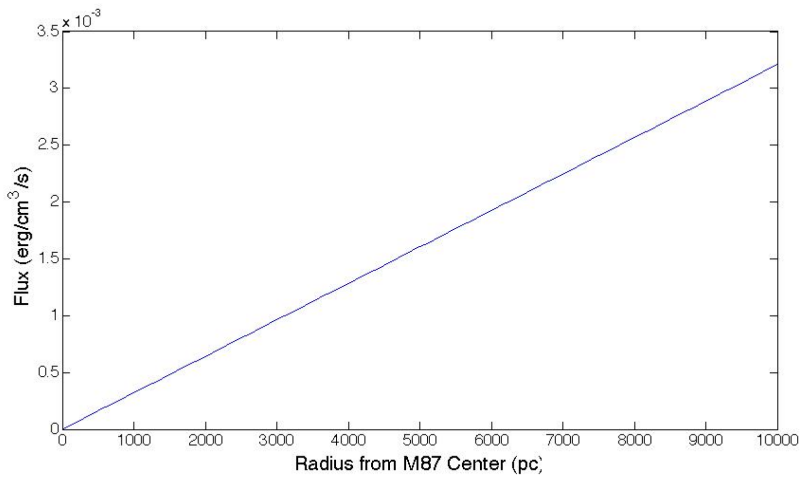


Figure 3: Energy density as a function of radius (in parsecs)

Using Wien's law to find the peak wavelength of a blackbody radiator:

$$\lambda_{max}T = 0.29cmK \approx 0.16cm \tag{49}$$

which is $\approx$ infrared

2 Problem 2

We study limb darkening by solving the radiation transport equation in plane parallel coordinates, assuming gray opacity:

$$\mu\frac{\partial I(\tau_z,\mu)}{\partial\tau_z} = I(\tau_z,\mu) - S(\tau_z,\mu). \tag{50}$$

We take as an ansatz that the angular dependence takes the form

$$I(\tau_z,\mu) = I_0(\tau_z) + I_1(\tau_z)\mu, \tag{51}$$

where

$$I_0 > I_1 > 0 \quad \forall \quad \tau_z.$$

c Mean intensity, flux, energy density, and radiation pressure

c.1 Mean intensity

In our coordinate system,

$$\mathrm{d}\Omega = \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi = -\,\mathrm{d}\mu\,\mathrm{d}\phi,$$

so

$$J = \frac{1}{4\pi}\oint I\,\mathrm{d}\Omega = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}(I_0 + I_1\mu)\,\mathrm{d}\mu\,\mathrm{d}\phi.$$

Evaluating the expression gives

$$J = I_0(\tau_z).$$

c.2 Flux

$$\begin{aligned} F &= \oint I \cos(\theta)\,\mathrm{d}\Omega \\ &= \int_0^{2\pi}\int_{-1}^{1}(I_0\mu + I_1\mu^2)\,\mathrm{d}\mu\,\mathrm{d}\phi. \\ &= \frac{4\pi I_1(\tau_z)}{3} \end{aligned}$$

c.3 Energy density

$$u(\tau_z) = \frac{4\pi}{c}J(\tau_z) = \frac{4\pi I_0(\tau_z)}{c}$$

c.4 Radiation pressure

$$p = \frac{1}{c}\int_0^{2\pi}\int_{-1}^{1}(I_0 + \mu I_1)\mu^2\,\mathrm{d}\mu\,\mathrm{d}\phi = \frac{4\pi I_0}{3c}.$$

The ratio of radiation pressure to energy density is 1:3, the same as for isotropic radiation.

d Integrating the zeroth moment of the radiation transport equation

The zeroth moment is simply Eq. (50) with $I(\tau_z)$ taking the form given in Eq. (51). This is integrated over all solid angles:

$$\begin{aligned} \mu\frac{\partial I}{\partial \tau} &= I - S \\ \oint \mu\frac{\partial}{\partial \tau_z}(I_0 + \mu I_1)\,\mathrm{d}\Omega &= \oint I_0 + \mu I_1 - S\,\mathrm{d}\Omega. \end{aligned}$$

We carry out the integration, making the assumption that $S \neq S(\mu)$, and find that

$$\frac{\partial}{\partial \tau_z}\frac{2}{3}I_1 = 2I_0 - 2S. \tag{52}$$

But

$$\frac{\partial}{\partial \tau_z}\left(\frac{2}{3}I_1\right) = \frac{1}{2\pi}\frac{\partial}{\partial \tau_z}F = 0,$$

since $F = \sigma T_{eff} \neq F(\tau_z)$. From earlier, we have the result that $J = I_0(\tau_z)$, so we can rewrite Eq. (52):

$$\begin{aligned} 0 &= 2J - 2S \\ \Rightarrow J &= S. \end{aligned}$$

e Integrating the first moment of the radiation transport equation

Multiplying the zeroth moment by μ and integrating over solid angles gives:

$$\int_0^{2\pi}\int_{-1}^{1}\mu^2\frac{\partial}{\partial\tau_z}\left(I_0+\mu I_1\right)\mathrm{d}\mu\,\mathrm{d}\phi=\int_0^{2\pi}\int_{-1}^{1}\mu I_0+\mu^2 I_1-\mu S\,\mathrm{d}\mu\,\mathrm{d}\phi,$$

which simplifies nicely:

$$\frac{\partial I_0}{\partial\tau_z}=I_1.$$

From a), we know that $I_1=\frac{3}{4\pi}F=\frac{3}{4\pi}\sigma T_{eff}^4$. Plugging this in yields:

$$I_0=\int I_1\,\mathrm{d}\tau_z=\frac{3}{4\pi}\sigma T_{eff}^4\tau_z+C,$$

where C is an integration constant. We can now write out an expression for the specific intensity up to a constant of integration:

$$I(\tau_z,\mu)=I_0+\mu I_1=\frac{3}{4\pi}\sigma T_{eff}^4\left(\tau_z+\mu+\tilde{C}\right).$$

f Determining the integration constant

Apply the approximation $F_{inward}|_{\tau_z=0}=0$. This gives:

$$\begin{aligned}0=\int_{inward}I\cos(\theta)\,\mathrm{d}\Omega &= \frac{3}{4\pi}\sigma T_{eff}^4\int_0^{2\pi}\int_{-1}^{0}\mu^2+\tilde{C}\mu\,\mathrm{d}\mu\,\mathrm{d}\phi\\ &= \frac{3}{2}\sigma T_{eff}^4\left(\left.\frac{\mu^3}{3}\right|_{-1}^{0}+\tilde{C}\cdot\left.\frac{\mu^2}{2}\right|_{-1}^{0}\right)\\ &= \frac{3}{2}\sigma T_{eff}^4\left(\frac{1}{2}-\frac{\tilde{C}}{2}\right)\\ \Rightarrow \tilde{C} &= \frac{2}{3}.\end{aligned}$$

We now have a full expression for the specific intensity:

$$I(\tau_z,\mu)=\frac{3}{4\pi}\sigma T_{eff}^4\left(\tau_z+\mu+\frac{2}{3}\right).$$

Plotted in Figure 4 is the emergent intensity:

$$\frac{I(\tau=0,\mu)}{I(\tau_z=0,\mu=1)}=\frac{3}{5}\left(\mu+\frac{2}{3}\right).$$

The atmosphere does appear to be radiative.

g Expression for the transit light curve

g.1 Approximations

1. We assume $R_p << R_s$

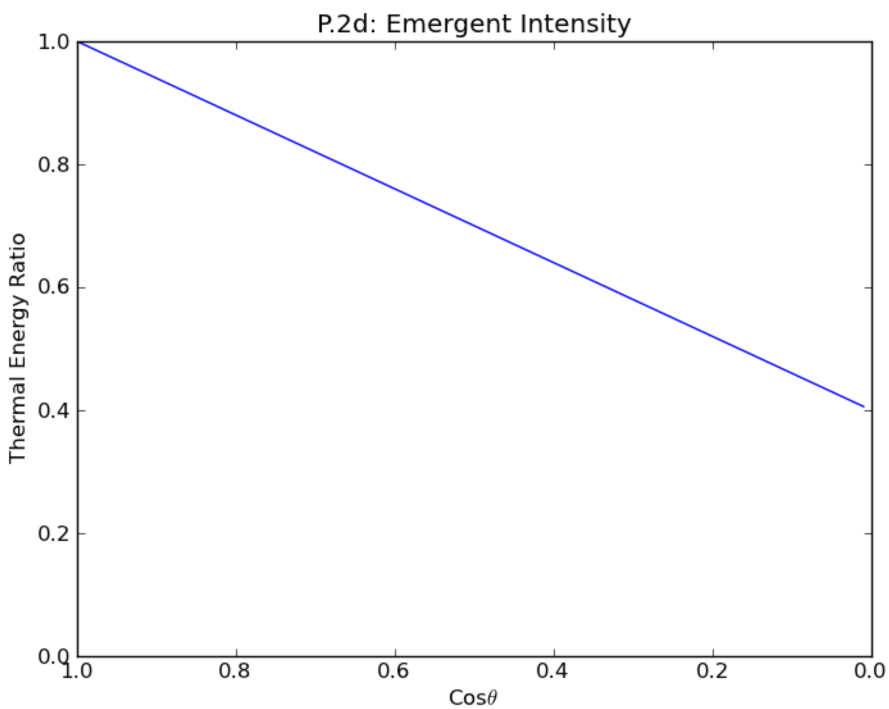


Figure 4: Limb darkening profile

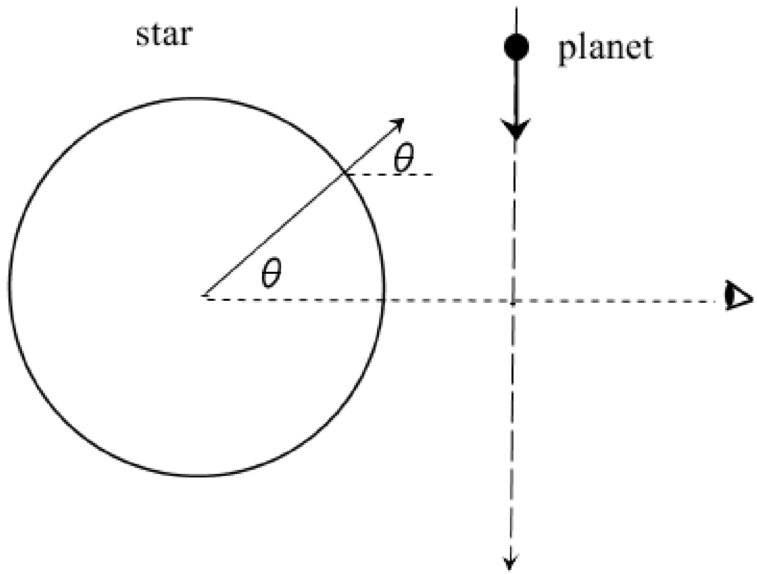


Figure 5: Diagram for the the transit light curve

2. Because the orbital radius is so much greater than the star, we approximate the planet's trajectory to be linear (i.e. $\frac{d}{dt}\sin(\theta)$, and not $\frac{d}{dt}\theta$ is constant, where θ measures the angle between the observer's line of sight and the line from the stellar center to the planet.) See Figure 1.

3. Because the observer is so far from the star/planet system, we take $\mu_{observer} = \cos(\theta)$. In other words, we take the angular variation over the surface of the star, relative to the observer, to be negligible. (See Figure 2.)

g.2 Flux due to the star

We calculate the total flux from the star without interference from the planet. Keeping in mind Approximation 3, we write:

$$\begin{aligned} F_0 &= \oint I(\mu)\cos(\alpha)\,\mathrm{d}\Omega \\ &= 2\pi \int I(\mu)\cos(\alpha)\,\mathrm{d}(\cos(\alpha)). \end{aligned}$$

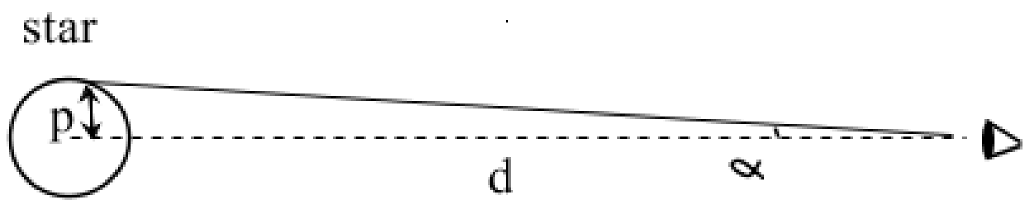


Figure 6: Diagram for calculating the flux of the star

Some basic geometry allows us to rewrite the integral:

$$\begin{aligned}
\cos(\alpha) = \frac{d}{\sqrt{d^2+p^2}} = \frac{1}{\sqrt{1+\frac{p^2}{d^2}}} &\simeq 1 - \frac{p^2}{2d^2} \\
\Rightarrow \cos(\alpha) &\simeq 1 \\
\Rightarrow \mathrm{d}(\cos(\alpha)) &\simeq \frac{p\,\mathrm{d}p}{2d^2}.
\end{aligned}$$

We also rewrite our expression for intensity:

$$\begin{aligned}
\mu(p) = \cos(\theta(p) + \alpha(p)) \simeq \cos(\theta(p)) &= \sqrt{1 - \left(\frac{p}{R_\star}\right)^2} \\
\Rightarrow I(\mu) \to I(p, \tau_z = 0) &= A\left(\frac{2}{3} + \sqrt{1 - \left(\frac{p}{R_\star}\right)^2}\right),
\end{aligned}$$

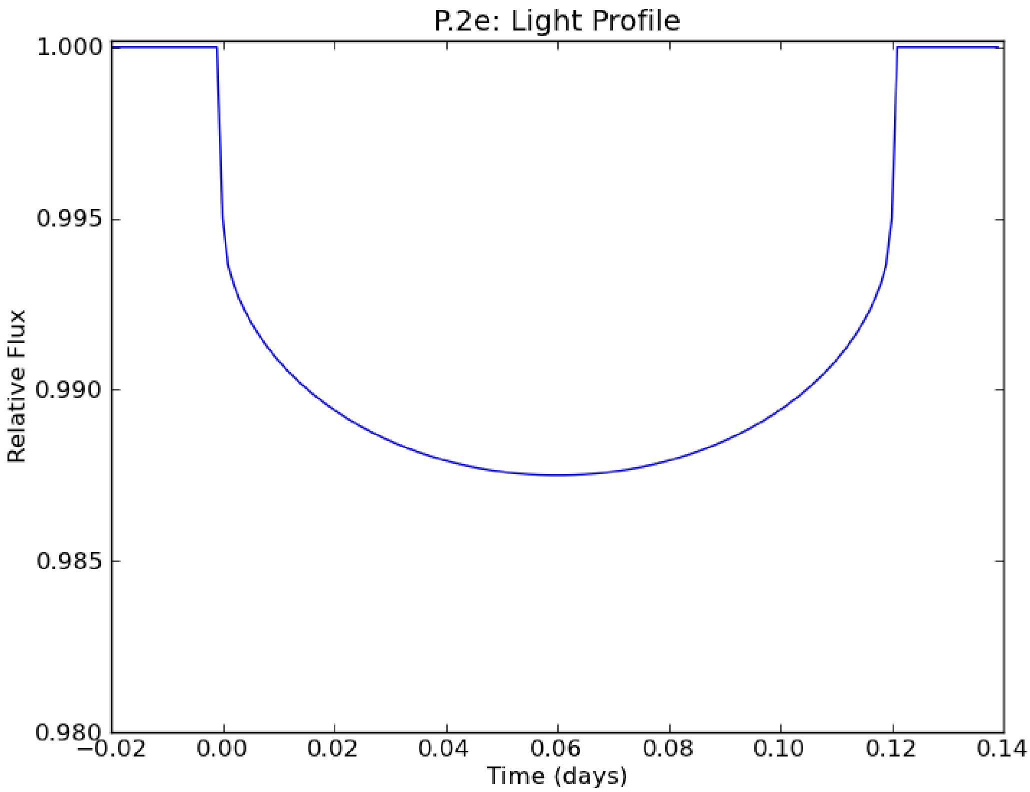
where A is a known constant (see Part d.) We are finally ready to calculate flux!

$$\begin{aligned}
F_0 &= \frac{2\pi A}{d^2}\int_0^{R_\star}\left(\frac{2}{3} + \sqrt{1 - \left(\frac{p}{R_\star}\right)^2}\right) p\,\mathrm{d}p \\
&= \frac{4\pi A}{3}\frac{R_\star^2}{d^2}
\end{aligned}$$

g.3 The flux blocked by the planet

It's an analogous setup, but we ignore the variation in I over the planet's surface.

$$\begin{aligned}
F_p &= 2\pi I(\mu)\int \cos(\alpha)(-\,\mathrm{d}\cos(\alpha)) = 2\pi I(p)\int_0^{R_p}\frac{p\,\mathrm{d}p}{d^2} \\
&= \frac{\pi A}{d^2}\left(\sqrt{1 - \frac{p^2}{R_\star}} + \frac{2}{3}\right) R_p^2.
\end{aligned}$$



g.4 Relative flux, as a function of time

The relative flux is given by:

$$F_{rel}(p) = \frac{F_p - F_0}{F_0} = 1 - \frac{3}{4}\left(\frac{R_p}{R_\star}\right)^2 \left(\sqrt{1 - \frac{p}{R_\star}^2} + \frac{2}{3}\right).$$

To convert to a function of time, we express impact parameter p as a function of t and the transit time T:

$$\begin{aligned} p(t) &= R_\star \,\text{abs}\left(1 - \frac{2t}{T}\right) \\ \Rightarrow F_{rel}(t) &= 1 - \frac{3}{4}\left(\frac{R_p}{R_\star}\right)^2 \left(\sqrt{\frac{4t}{T}\left(1 - \frac{t}{T}\right)} + \frac{2}{3}\right). \end{aligned}$$

To plot this, we take $T = .12$ days, and $\frac{R_p}{R_\star} \simeq \frac{R_{Jupiter}}{R_\odot} \simeq .1$. The plot looks reasonable, given our simplifications.

h The sun's red edge

We make the additional assumption that the atmosphere everywhere is in local thermal equilibrium, so $S(\tau) = B(\tau)$. The radiation transport equation for LTE is:

$$\mu \frac{\partial I}{\partial \tau_z} = I - B(T).$$

Plugging in our expressions for I and $B(T)$ gives:

$$T(\tau_z) = T_{eff} \left(\frac{3}{4}\tau_z + \frac{1}{2} \right)^{\frac{1}{4}}.$$

We assume that we "see" the temperature at a line of sight optical depth $\tau = \frac{2}{3}$. At the center of the sun, $\mu = 1$. The line of sight is radially inward, so $\tau_z = \tau = \frac{2}{3}$.

$$\rightarrow T\left(\frac{2}{3}\right) = T_{eff} \times 1 = 5800\,\mathrm{K}.$$

At the edge of the sun, $\mu = 0$. We take $\tau_z \simeq 0$, so

$$T(0) = T_{eff} \left(\frac{1}{2}\right)^{\frac{1}{4}} \simeq 4880\,\mathrm{K},$$

which is almost 1000 K cooler, thus explaining why the the edge of the sun appears redder.